Sperner’s Lemma
Brouwer’s Fixed Point Theorem
The Fundamental Theorem of Algebra

Larry Taylor
ltaylor@csubak.edu
http://www.cs.csubak.edu/~larry/sperner.pdf

<u>Notation</u>: T denotes a triangle with vertices $\mathbf{p}^0$, $\mathbf{p}^1$, $\mathbf{p}^2$

<u>Comments</u> $\mathbf{x} \in$ T iff $\mathbf{x} = \sum x_i \mathbf{p}^i$ where $0 \le x_i \le 1$, $\sum x_i = 1$. The x_i are uniquely determined and are the <u>barycentric coordinates of **x**</u>. The set of points where one (or more) coordinate is 0 is called a <u>face</u> of T.

<u>Sperner’s Lemma</u> (two-dimensional version). Triangulate T. Give each vertex of the triangulation a label from {0,1,2} in such a manner that a lower dimensional face of T not containing $\mathbf{p}^i$ (i.e for which the i-th barycentric coordinate is 0) is not labelled i. (Thus, each $\mathbf{p}^i$ is labelled i.) <u>Then</u> there is a completedly labelled triangle, i.e with labels all of {0,1,2}.

<u>Brouwer’s Fixed Point Theorem</u> (two-dimensional version) Assume f: T -> T is continuous. <u>Then</u> f has a fixed point.

<u>Sperner -> Brouwer</u> Let f:T -> T be continuous with no fixed point. Label each $\mathbf{x} \in$ T with an $i \in \{0,1,2\}$ so that $f(\mathbf{x})_i < x_i$. (loosely, f maps **x** "away" from $\mathbf{p}^i$) Note that any triangulation of T will satisfy the hypotheses of Sperner’s lemma.

For each $k \in \mathbf{N}$, let G_k be a triangulation with diameter of subtriangles < 1/k. Each triangulation must contain a completely labeled triangle. Let **y** be an accumulation point of these completely labelled triangles. For each i, there are points **x** arbitrarily close to **y** where $f(\mathbf{x})_i < x_i$. By continuity, $f(\mathbf{y})_i \le y_i$ for all i. But the coordinates of any point must sum to 1. So $f(\mathbf{y})_i = y_i$ for each i and this means $f(\mathbf{y}) = \mathbf{y}$. Contradiction.

<u>Oriented Sperner’s Lemma</u> (a two-dimensional version). If T is triangulated arbitrarily then: the following are equal:

$\#({}_0{}^2{}_1) - \#({}_1{}^2{}_0)$	=	$\#(0\ 1) - \#(1\ 0)$
(counted over all triangles)		(counted counterclockwise over boundary)
N[T]	=	N[bd(T)]

(Notation to be explained later. Easy to check this does generalize the previous Sperner’s lemma)

Definition. An (oriented) n-simplex is an ordered (n+1)-tuple of points in $\mathbf{R}^m$ for some m, and is denoted: $<\mathbf{p}^0,\mathbf{p}^1,...\mathbf{p}^n>$.

Definition. Two simplices are equal if they consist of the same points and one is an even permutation of the other. A k-chain is an integer linear combination of k-simplices. We identify an odd permutation of the points of a simplex with minus the simplex.

e.g. $<\mathbf{p}^0, \mathbf{p}^1> = -<\mathbf{p}^1, \mathbf{p}^0>$

$<\mathbf{p}^0, \mathbf{p}^1, \mathbf{p}^2> = <\mathbf{p}^2, \mathbf{p}^0, \mathbf{p}^1> = - <\mathbf{p}^1, \mathbf{p}^0, \mathbf{p}^2>$

Definition. (boundary of a simplex)

$$\mathrm{bd}(<\mathbf{p}^0, \mathbf{p}^1>) = <\mathbf{p}^1> - <\mathbf{p}^0>$$

$$\begin{aligned}\mathrm{bd}(<\mathbf{p}^0, \mathbf{p}^1, \mathbf{p}^2>) &= <\mathbf{p}^1, \mathbf{p}^2> - <\mathbf{p}^0, \mathbf{p}^2> + <\mathbf{p}^0, \mathbf{p}^1> \\ &= <\mathbf{p}^1, \mathbf{p}^2> + <\mathbf{p}^2, \mathbf{p}^0> + <\mathbf{p}^0, \mathbf{p}^1>\end{aligned}$$

(More generally, $\mathrm{bd}(<\mathbf{p}^0,...\mathbf{p}^k>) = \Sigma_j(-1)^j<\mathbf{p}^0,..\mathbf{p}^{j-1},\mathbf{p}^{j+1}..\mathbf{p}^k>$)

The boundary of a chain is defined by extending linearly:

if $C = \Sigma n_i S_i$, then $\mathrm{bd}(C) = \Sigma n_i \mathrm{bd}(S_i)$.

Note how the boundary respects orientation and how we can construct a chain of simplices of the same dimension getting the intuitive notion of boundary (because of cancellation of interior boundaries).

Definition. Let $q_0,..q_k$ be non-negative integers. If $q_0,...,q_k$ is a permutation of 0,..,k then $N(q_0,..q_k)$ is ±1 depending on its sign. Otherwise it is zero.

$N(0,1,2) = N(2,0,1) = N(0,1) = N(0) = 1$

$N(0,2,1) = N(1,0) = -1,\ N(0,0,1) = N(0,1,3) = N(0,2) = N(1) = 0$

Suppose each vertex **p** of a simplex is given a label $q(\mathbf{p})$. Define $N(<\mathbf{p}^0,...\mathbf{p}^k>) = N(q(\mathbf{p}^0),...q(\mathbf{p}^k))$. Extend N linearly to chains.

Theorem (Oriented form of Sperner's Lemma) Let C be any k-chain with the vertices of simplices of C having only labels chosen from 0,...k. Then

$$N[C] = (-1)^k N[\mathrm{bd}(C)].$$

e.g. If C is a 1-chain: $\#(0,1) - \#(1,0) = - N(\mathrm{bd}(C))$

This is a generalization of Sperner's lemma in the case when C is a correctly labelled n-simplex. For, given a labelling as in the hypotheses of Sperner's lemma, let F^k be the k-face with corners labelled 0,...k. Then $N(C) = N(F^n) = \pm N(F^{n-1}) = \pm N(F^{n-2}) = ... = \pm N(F^0) = \pm 1$.

Proof. We need only consider a k-simplex, $s^k = \langle \mathbf{p}^0, ... \mathbf{p}^k \rangle$. The result is is easy to check directly by cases for k = 1 or 2. In general, the claim amounts to showing: If $q_0, .. q_k \in \{0, ..., k\}$

$$N(q_0, ... q_k) = (-1)^k \Sigma (-1)^j (q_0, .. q_{j-1}, q_{j+1}, .. q_k).$$

Let M be the "permutation" matrix defined by $(q_0, ... q_k)$:

$m_{ij} = 1$ if $q_i = j$
$m_{ij} = 0$ otherwise

The left hand side above is det(M). Let M* be M with the last column replaced by ones. The right hand side is det M* (expand by last column). We can reduce M to M* by adding all other columns of M to the last column. So det(M) = det(M*).

To prove the fundamental theorem of algebra, we begin by dividing the complex plane into three "tridrants" by letting

$$R_j = \{z \in \mathbf{C} \mid 2\pi j/3 \leq \arg(z) < 2\pi(j+1)/3\} \text{ for } j = 0,1,2.$$

A geometric argument shows:

Lemma. Suppose $z_j \in R_j$ for j = 0,1,2 and $|z_j - z_k| < \in$ for j,k = 0,1,2. Then $|z_j| < 2\in/\sqrt{3}$ for j = 0,1,2.

Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ be a non-constant complex polynomial. $\exists y \in \mathbf{C}$ with $p(y)=0$.

Proof. Give any point z the label j corresponding to the region R_j in which p(z) lies. If we let T be a large enough disk, then on its boundary $p(z) \sim z^n$. Thus, if we have a fine triangulation G, of T, then N(bd(G)) = n.

For each $k \in \mathbf{N}$, let G_k be a triangulation of G with triangles of diameter < 1/k. It follows from the lemma that $|p(z_j)| < 2/(\sqrt{3}k)$ for z_j a vertex of a completely labelled triangle of G_k. Thus, if y is an accumulation point of these triangles, there are points z arbitrarily close to y where |p(z)| is arbitarily small. Therefore, by continuity |p(y)| = 0.

(The preceding is a discrete version of the winding number proof.)