Maple syntax can sometimes be tricky. Suppose that we define a variable \( x \) to it, and define another variable \( q \) to be an expression involving \( x \)
\[
\begin{align*}
x &:= 5; \\
q &:= x^2 - (10/3)x + 2;
\end{align*}
\]
will then show the answer \( 31/3 \), as it should. But if we now ask to solve for when \( q = 0 \) by typing
\[
solve(q=0);
\]
we do not get any answer. The problem is that \( x \) is no longer an independent variable – we assigned it the value 5. To remedy this situation we need to do
\[
\begin{align*}
\text{readlib(unassign);} \\
\text{unassign('x');} \\
q &:= x^2 - (10/3)x + 2; \\
solve(q=0);
\end{align*}
\]
and we get the correct exact answers
\[
\frac{5}{3} + \frac{1}{3}\sqrt{7}, \frac{5}{3} - \frac{1}{3}\sqrt{7}
\]
To avoid these pitfalls, I would recommend that you not assign simples variables \( x, y, \) and \( z \), numerical values and that you not use expressions like \( q \) but instead use the function notation we introduced last time. For example,
\[
\begin{align*}
f &:= x \to x^2 - (10/3)x + 2; \\
solve(f(x)=0,x);
\end{align*}
\]
produces the same roots as above. Note that the \texttt{solve} function, when used with a function rather than an expression, requires as a second parameter the variable we want to find the solution in terms of. Note that we can convert the two roots to floating point approximations by using \texttt{fsolve} instead of \texttt{solve} as follows
\[
\begin{align*}
\text{fsolve}(f(x)=0,x);
\end{align*}
\]
which yields the decimal approximations \( 0.7847495630 \) and \( 2.548583770 \).
It is possible to obtain closed form solutions for roots of polynomials of
degree less than or equal to four. For fifth degree and higher polynomials
there are no such algorithms. Newton’s method is a way to use the deriva-
tive and repeated estimation to find accurate, but approximate, values for
the roots. To try this, let’s first plot the function $f$ above with a carefully
chosen tangent line. We need to use the `showtangent` command and so we
need to load

```maple
with(plots);
with(student);
```

Ignore the warning message because we want $D$ to be redefined. It is clear
by looking at a graph of $f$

```maple
plot(f(x),x=-4..4);
```

that there is one root to the right of 0 and another root to the left of 3.
Let’s plot the curve with these tangent lines

```maple
showtangent(f(x),x=0,x=-1..4,y=-3..6);
showtangent(f(x),x=3,x=-1..4,y=-3..6);
```

where we have restricted the range of $y$ so that the graphs are more read-
able. It is clear that where the tangent lines intersect the $x$-axis we have an
approximation for the roots. A little algebra (you should check this) shows
that this point is

$$x - \frac{f(x)}{f'(x)}$$

where $x$ is (respectively) 0 and 3. We could therefore define

```maple
newton := x -> evalf(x - (f(x)/D(f)(x)));
```

so that `newton(0)` and `newton(3)` are the two approximations. If we repeat
this approximation we can do better. For example, trying

```maple
newton(newton(0));
newton(newton(newton(0)));
newton(newton(newton(newton(0))));
```

gives closer and closer approximations to the actual root (we could have
used the `%` symbol to save some typing here). And we could repeat the
calculation with `newton(3)`.

A better solution would be to use a programming loop. We first have to
decide how accurate we want the answer to be. Say we want it to be within
$10^{-8}$. We then define a tolerance function

```maple
tol := (x,y) -> abs(x - y);
```

and write the following loop
a := 0;
aold := a;
a := newton(a);
for n from 1 to 20 while (tol(a,aold) > 10^(-8)) do:
   aold := a:
a := newton(a):
od;

The syntax here is very tricky. First note that we have used colons : instead of semicolons ; except for the last line of the loop. This is to prevent Maple from giving us error messages about incomplete statements. Second, we have used a and aold to denote successive approximations, so we are going to stop the when two successive approximations are within 10^{-8} of one another (or when we have gone through the loop 20 times). Run this loop and see that the loop stops when the desired accuracy is obtained. Then do the same thing starting with a := 3 to find an approximation for the other root.

**Assignment:**

1. Define the following function

   \[ g(x) = x^2 + \frac{1}{x} - \frac{1}{x^2} \]

   and use `fsolve` to find an approximate solution to

   \[ x^2 + \frac{1}{x} - \frac{1}{x^2} = 0 \]

   The approximate solution obtained should be negative. Plot the function separately on the intervals (-10, -0.01) and (0.01, 10) (why?) and note that there is a second solution which is positive. Find an approximation to the positive solution by using the modification to the `fsolve` command of the form

   `fsolve(g(x)=0,x,x=A..B);`

   where Maple limits its search to the interval \((A, B)\).

2. Use the function `newton` with the polynomial \( f(x) = x^2 - 5 \) starting with first point \( x = 0 \). What happens? Explain Maple’s error message after you have plotted a graph of this polynomial together with its tangent line at \( x = 0 \) for insight.
3. The polynomial \( p(x) = x^3 + 94x^2 - 389x + 294 \) has roots at 1, 3, and \(-98\). The point \( x = 2 \) might therefore be thought to yield a good starting point for finding either of the smaller roots. Try this and explain what happens.