Mathematics 222 – Module 4
Some Fine Points of Maple
and Numerical Integration
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Generally, unless you are only going to do very simple work it is best to start by loading the student and plots packages

\[
\text{with(student);}
\]
\[
\text{with(plots);}
\]
and loading the unassign function

\[
\text{readlib(unassign);}
\]
If you plan to re-run the worksheet many times it is best to unassign variables at the beginning of the worksheet (otherwise the second attempt to run may fail). For the next example I am going to use the variables \( x \) and \( \theta \) so I will do

\[
\text{unassign('x','\theta');}
\]
Note the single quotes around each variable. Whereas, \( x \) means the evaluation (if any) of \( x \), \( 'x' \) means \textit{literally} \( x \).

A primary problem with maple is that when a variable is reused in a (new) function definition, it may have some context from a previous use. Try the following. Define \( f \) to be the squaring function and check the definition.

\[
f := x \rightarrow x^2;
\]
\[
f(x);
\]
Now define \( x \) to depend on \( \theta \) and recheck the definition of \( f \).

\[
x := \tan(\theta);
\]
\[
f(x);
\]
\[
diff(f(x),x);
\]
\[
diff(f(x),\theta);
\]
Note that \( x \) is no longer an independent variable and that \( \text{diff}(f(x),x) \) fails because Maple evaluated \( f \) in terms of \( x \) and \( x \) in terms of \( \theta \), so that as far as the differentiation operator is concerned, \( x \) is no longer there.
But if we now change the dependency of \( x \) on \( \theta \) we see that this changes \( f \) once more

\[
x := \theta^2;
\]
\[
f(x);
\]
\[
diff(f(x),\theta);
\]
It is easy to see what is happening if everything is all on one page, but if
the worksheet is 10 pages long, a previous dependency may be forgotten. If
we don’t want \( f \) to depend on \( \theta \) we must do the following

\[
\text{unassign(’x’);} \\
\text{f(x);} \\
\text{Now checking derivatives shows that there is no dependency of } f \text{ on } \theta:\n\]

\[
\text{diff(f(x),x);} \\
\text{diff(f(x),\theta);} \\
\]

We saw last time that derivatives can be approximated numerically. It is
even more common that definite integrals, such as \( \int_a^b g(x)dx \) are evaluated
numerically, since not all elementary functions have elementary antiderivatives. For example, consider

\[
g := x \rightarrow (x + 1)/x; \\
\text{Find an antiderivative for } g(x) \text{ by hand. Check your result with}
\]

\[
\text{int(g(x),x);} \\
\text{Suppose that we want to find the definite integral of } g(x) \text{ from 1 to 5:}
\]

\[
\text{int(g(x),x=1..5);} \\
\text{evalf(\%);} \\
\text{gives us the answer (5.609437912) to 10 significant digits. Maple allows us}
\]

\[
\text{to do Riemann Sums with (alternatively) the left or right endpoint as } x_i^*.
\]

Let’s use a subdivision of the interval \([1,5]\) into 10 equal parts:

\[
\text{leftbox(f(x),x=1..5,10);} \\
\text{leftsum(f(x),x=1..5,10);} \\
\text{evalf(\%);} \\
\text{rightbox(f(x),x=1..5,10);} \\
\text{rightsum(f(x),x=1..5,10);} \\
\text{evalf(\%);} \\
\]

Why did the left sum overestimate? Why did the right sum underestimate?
In general, the accuracy of the left (or right) sum depends on how large
\( g'(x) \) is. It can be proved that the error \( e_{rs} \) is bounded by

\[
e_{rs} \leq \frac{M(b-a)^2}{2n}
\]

where \( M = \max\{|g'(x)| : a \leq x \leq b\} \), and \( n \) is the number of subdivisions
of \([a,b]\). Compute this upper bound for the error with Maple:

\[
\text{M := maximize(abs(D(g)(x)),x,{x=1..5});} \\
\text{error_{rs} := (n,a,b) \rightarrow (M * (b - a)^2) / (2*n);} \\
\]
\texttt{error}_rs(10,1,5);
\texttt{evalf(\%);}  
You should get 0.8 (which is not that good, but we only used \( n = 10 \)).

\textbf{Assignment:}  

1. Using the same function \( g(x) = (x+1)/x \) above find \( n \) sufficiently large so that a left (or right) sum is within 0.0001 of the actual integral \( \int_1^5 g(x) dx \). Note that you can use the \texttt{solve} command in Maple as follows:

\[
\texttt{solve(error}_rs(n,1,5) = .0001,n);
\]

Does this large value surprise you?

2. A better method for approximating definite integrals is \textit{Simpson’s rule}. This method essentially replaces short segments of \( g(x) \) with pieces of a parabola. In this case the error \( e_{simp} \) can be shown to be bounded by

\[
e_{simp} \leq \frac{M(b - a)^{5}}{180n^{4}}
\]

where \( M = \max\{|g^{(4)}(x)| \mid a \leq x \leq b\} \), and \( n \) is the number of subdivisions of \([a,b]\). Compute the value Simpson’s rule gives with Maple using

\[
\texttt{simpson(g(x),x=1..5,10); evalf(\%);} 
\]

and give an upper bound for the error. Note that you will need to compute the maximum of the absolute value of the fourth derivative of the function on \([1,5]\).

3. Find \( n \) sufficiently large so that Simpson’s rule yields a value within 0.0001 of the actual integral \( \int_1^5 g(x) dx \) and find the actual estimate (note that you will have to round up to the nearest even integer because Simpson’s rule requires \( n \) to be even). Given the same desired accuracy, which method, right hand sums or Simpson’s rule, would be most likely to require the smaller value of \( n \)?

4. Let \( h \) be the following function

\[
h(x) = \frac{1}{\sqrt{2\pi}}e^{-(x^2/2)}
\]

This is actually the probability density function for a standard
normal distribution with mean zero and variance one. Estimate

\[ \int_{-2}^{2} h(x) \, dx \]

using Simpson’s rule with a sufficiently large \( n \) so that the error is less than 0.0001 of the actual value.