Let \( \mathbb{N} \) denote the natural numbers, \( \mathbb{Q} \) denote the field of rational numbers, \( \mathbb{R} \) denote the field of real numbers, and \( \mathbb{C} \) denote the field of complex numbers, so

\[
\mathbb{Q} \equiv \{ \pm p/q \mid p, q \in \mathbb{N} \text{ and } q \neq 0 \}
\]

\[
\mathbb{C} \equiv \{ a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1 \}
\]

\( \mathbb{R} \) is harder to define but we can identify each real number with an infinite decimal which does not eventually become all 9’s (why?). It is clear that \( \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \).

1. Show that the rational numbers correspond precisely to those real numbers which have repeating decimals (i.e. eventually there is a finite pattern which repeats indefinitely).

2. For each of the statements below do one of the following: prove that \( y \) is rational, prove that \( y \) is irrational, or conclude that there is not enough evidence to decide.

2a. \( x \) is rational and \( x + y \) is rational.

2b. \( x \) is rational and \( xy \) is rational.

2c. \( x \) is rational, \( x \neq 0 \) and \( xy \) is rational.

2d. \( x \) is irrational and \( x + y \) is irrational.

2e. \( x \) is irrational and \( x - y \) is rational.

3. A field \( \mathbb{F} \) is called algebraically closed if every polynomial equation with coefficients from the field has at least one root, i.e., if \( a_i \in \mathbb{F} \) for \( i = 0, 1, \ldots, n \) then

\[
\alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_1 x + \alpha_0 = 0
\]

has at least one solution in \( \mathbb{F} \). Which of \( \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) are algebraically closed (you will only be able to prove your answer for two of the above)?

4. Not all fields are infinite sets. We could let \( p \) be any prime number in the following but for now take \( p = 5 \) and let \( \mathbb{Z}_5 \equiv \{ 0, 1, 2, 3, 4 \} \). Addition, subtraction, and multiplication are performed \( \text{mod } 5 \). For example, \( 3 \cdot 4 = 12 \) but dividing 12 by 5 leaves a remainder of 2 so we write \( 3 \cdot 4 \equiv 2 \). Make a multiplication table for \( \mathbb{Z}_5 \). From the multiplication table show that for every non-zero element \( x \in \mathbb{Z}_5 \) there exists \( y \in \mathbb{Z}_5 \) such that \( xy \equiv 1 \). Define \( y = 1/x \) and you have division (why?).

5. What fails in problem 4. if you don’t use a prime number? For example, why is \( \mathbb{Z}_4 \equiv \{ 0, 1, 2, 3 \} \) not a field?

6. Define \( \mathbb{F} \equiv \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \) Show that \( \mathbb{F} \) is a field between \( \mathbb{Q} \) and \( \mathbb{R} \). Note that the primary difficulty is showing that every non-zero element \( x \) has a reciprocal in \( \mathbb{F} \), i.e., there is some \( y \in \mathbb{F} \) such that \( xy = 1 \).