1. Find the following limits using any theorem that we have proved, or, if necessary, by using the the $\epsilon-N$ definition of the limit of a sequence:

1a. $\lim_{n \to \infty} (6n)^1/(2n)$

1b. $\lim_{n \to \infty} (10n^2 - 4)/(n^3 + 5n^2 + 1)$.

1c. $\lim_{n \to \infty} (1 + 1/n^2)^{n^2}$.

2. Let $k \geq 1$ be fixed in $\mathbb{N}$. Show that for any $n \in \mathbb{N}$

$$
\frac{1}{(n+1)^{k+1}} \leq \frac{1}{kn^k} - \frac{1}{k(n+1)^k}
$$

3. Use the result in Problem 2 to show that the sequence

$$
\left\{ x_n = \frac{1}{1^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{n^{k+1}} \right\}_{n=1}^{\infty}
$$

is Cauchy for $k \in \mathbb{N}$. Note the result fails if $k = 0$, Why? Also note since the sequence is Cauchy for $k \in \mathbb{N}$ it is convergent, but it is not at all clear what the limit actually is.

4. Let $\{x_n\}_{n=1}^{\infty}$ be a listing of all of the rationals in $[0, 1]$ (we can do this because the set is countable). Let $\{y_n\}_{n=1}^{\infty}$ be a new sequence defined as follows:

$$
y_n = \frac{1}{q} \quad \text{if} \quad x_n = \frac{p}{q}
$$

where $p$ and $q$ share no common factors (i.e. the fraction for $x_n$ is “reduced”). Why is every subsequence of $\{y_n\}_{n=1}^{\infty}$ convergent? What is the limit in each case?

5. Give an example of a sequence $\{x_n\}_{n=1}^{\infty}$ which is not Cauchy but satisfies

$$
\lim_{n \to \infty} |x_{n+1} - x_n| = 0
$$

Let $k > 1$ be fixed in $\mathbb{N}$. Give an example of a sequence which is not Cauchy but satisfies

$$
\lim_{n \to \infty} |x_{n+k} - x_n| = 0
$$

6. Let $\{x_n\}_{n=1}^{\infty}$ be a contractive sequence, i.e. there exists $0 < C < 1$ such that

$$
|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|
$$

for $n = 1, 2, 3, \ldots$. Find an upper bound for $|x_n|$ in terms of $C$, $x_1$, and $x_2$. 