MATH 402 – LAB 7

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1. Two functions $u(x, y)$ and $v(x, y)$ on $\mathbb{R}^2$ are said to be harmonic conjugates if they satisfy the (Cauchy-Riemann) equations $u_x = v_y$ and $u_y = -v_x$. Show that $u(x, y) = \cosh y \cos x$ and $v(x, y) = -\sinh y \sin x$ are harmonic conjugates (recall that $\cosh y = (e^y + e^{-y})/2$ and that $\sinh y = (e^y - e^{-y})/2$).

2. Show that if $u(x, y)$ and $v(x, y)$ are harmonic conjugates then

$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0,$$

so the functions themselves are actually harmonic in that they satisfy Laplace’s equation.

3. Show that $f(x + iy) = \cosh y \cos x - i \sinh y \sin x$ is a complex function of a complex variable which equals the cosine function on the real axis. Consequently, it is the natural extension of the cosine function $\cos x$ to complex numbers $x + iy$.

4. Let $i$ denote the imaginary number such that $i^2 = -1$ and let $x$ be any real number. Show that

$$\left| \frac{2x}{x^2 + 1} + i \left( \frac{x^2 - 1}{x^2 + 1} \right) \right| = 1$$

where $|a + ib| = \sqrt{a^2 + b^2}$. Using this, then show that $(x - i)/(ix + 1)$ is on the unit circle in the complex plane (hint: multiply by $(ix + 1)/(ix + 1)$).

5. Define the analytic mapping

$$f(z) = \frac{z - i}{-iz + 1},$$

where $z = x + iy$. Why does this function map the real axis onto the unit circle in the complex plane? Since the function is continuous it must either map the upper half plane \{ $x + iy \mid y > 0$ \} entirely inside the unit disk or entirely outside the unit disk. Which of these two possibilities occurs?