§0. Notation and Euclidean Space. There are two common ways to denote vectors in the Euclidean Space $\mathbb{R}^n$. One is to designate a standard orthogonal unit vector basis $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ and write a general vector $\vec{v}$ as a linear combination

$$\vec{v} = v_1\hat{x}_1 + v_2\hat{x}_2 + \ldots + v_n\hat{x}_n$$

The second is to simply list the unique coefficients in the linear combination above as an n-tuple

$$\vec{v} = (v_1, v_2, \ldots, v_n)$$

For the following theory, one doesn’t need any of the special coordinate systems (cylindrical, spherical, etc.) so I will primarily use this latter form of notation.
§1. Manifolds and Orientability. In ordinary words, an \( n \)-dimensional manifold is a topological object which has been built by “gluing” together patches from some Euclidean space \( \mathbb{R}^n \), and which is “smooth” except perhaps for some “folding” or “creasing” at the glue lines.

To be more specific, a 1-dimensional manifold is a piecewise smooth curve in \( \mathbb{R}^n \) such that each smooth curve segment can be parametrized as a mapping from some closed interval \([a, b]\) to \( n \)-dimensional Euclidean space:

\[
\vec{r}(u) : [a, b] \to \mathbb{R}^n,
\]

and such that each component of the vector function

\[
u \to \vec{r}(u) = (x_1(u), x_2(u), \ldots, x_n(u))
\]

is differentiable for \( u \in (a, b) \) and continuous for \( u \in [a, b] \). Notice that there is a natural orientation on the curve segment whenever a parametrization is given; we “start” at \((x_1(a), x_2(a), \ldots, x_n(a))\) and “stop” at \((x_1(b), x_2(b), \ldots, x_n(b))\). We could always reverse this, starting at \( u = b \) and going back to \( u = a \). Although a parametrization is not unique, in the case of a rectifiable curve segment it is always possible to (re)parametrize with respect to arc length \( s \)

\[
s \to \vec{r}(s) = (x_1(s), x_2(s), \ldots, x_n(s))
\]

for \( s \in [0, L] \) where \( L \) is the total length of the curve. This parametrization is unique once a direction on the curve has been chosen. If the curve is built from more than one curve segment, we insist that the individual orientations be compatible, that is, the stopping point of the \( k \)-th segment should be the same as the starting point of
the \((k+1)\)-st segment, and if the curve is a closed curve, then the last stopping point should be the first starting point.

The advantage of a parametrization with respect to arc length \(s\) is that we have a unit-speed curve. Consequently the unit-tangent vector function \(\vec{t}(s)\) can be easily computed as

\[
\vec{t}(s) = \frac{d\vec{r}(s)}{ds} = \left( \frac{dx_1(s)}{ds}, \frac{dx_2(s)}{ds}, \ldots, \frac{dx_n(s)}{ds} \right)
\]

since

\[
\left\| \frac{d\vec{r}(s)}{ds} \right\| = 1
\]

Because the 1-form \(\vec{t}(s)ds\) occurs so often when doing line integrals, many books use the shorthand notation

\[
d\vec{s} = \vec{t}(s)ds
\]

Whenever we have an \(n\)-dimensional oriented manifold \(S\) the boundary \(\partial S\) is automatically an \((n-1)\)-dimensional oriented manifold.

For the smooth curve segment \(C\) with unit-speed parametrization \(\{\vec{r}(s)\}_{s=0}^{L}\) above, the (oriented) boundary \(\partial S\) consists of the two endpoints with \(-1\) attached to the starting point \(\vec{r}(0)\) and \(+1\) attached to the ending point \(\vec{r}(L)\). This \pm 1 weighting is necessary to preserve both orientational and numerical consistency. Since the stopping point of the \(n\)-th segment is weighted with \(+1\) and equals the starting point of the \((n+1)\)-st segment which is weighted with \(-1\), the weights will cancel unless we are dealing with the actual initial and final points of the curve (which, if the curve is closed, will also cancel, and we will have a closed loop with no boundary points). In addition, if we reverse the direction on the curve, the value of a line integral \((\vec{A}\) is
some vector field
\[ \int_{C} \vec{A} \cdot d\vec{s} = \int_{C} \vec{A} \cdot \vec{t}(s) ds \]
will be negated (as it should be).

In a corresponding way, a \textit{2-dimensional manifold} is built by “gluing” together “smooth” surface patches along their boundaries. Each \textit{surface patch} can be parametrized as a mapping from some closed region \( R \) (with no “holes”) of \( \mathbb{R}^2 \) to \( n \)-dimensional Euclidean space:

\[ \vec{r}(u, v) : R \to \mathbb{R}^n , \]

and such that each component of the vector function

\[ (u, v) \to \vec{r}(u, v) = (x_1(u, v), x_2(u, v), \ldots, x_n(u, v)) \]

is differentiable for \((u, v) \in \text{interior}(R)\) and continuous for \((u, v) \in R\). It is not so clear what the natural \textit{orientation} for a surface patch should be, or even how a surface can be oriented. At this point it is best to consider an example.

It is routine to check that a \textit{sphere} of radius \( a \) sitting in \( \mathbb{R}^3 \) can be parametrized with a function of two variables (using a single surface patch) with the first variable \( u \) playing the role of “longitude” (measured in the counterclockwise direction from the positive \( x \)-axis) and the second variable \( v \) playing the role of “latitude” measured up or down from the \( xy \)-plane. Just let \( R \) be the rectangle

\[ R = \{(u, v) \mid 0 \leq u \leq 2\pi \text{ and } -\pi/2 \leq v \leq \pi/2\} , \]

and define

\[ \vec{r}(u, v) = (a \cos u \cos v, a \sin u \cos v, a \sin v) \]

for \((u, v) \in R\) Orientation should be a “standard” or “assignment” which enables any inhabitant of the manifold to tell \textit{right-handed} objects from \textit{left-handed} ones. For example, our real \textit{3-dimensional} world
has, over centuries, developed an orientation (not all that enlightened, since the word for “left” in most early languages also has a pejorative connotation, c.f. “sinister” from the Latin word sinestre for “left”). Telling left-handed from right-handed is equivalent to being able to unambiguously decide whether a local coordinate system in the manifold is left or right handed. But with a parametrization we always have a natural local coordinate system by using the partial derivatives; in this example

\[
\frac{\partial \vec{r}}{\partial u} = (-a \sin u \cos v, a \cos u \cos v, 0)
\]

\[
\frac{\partial \vec{r}}{\partial v} = (-a \cos u \sin v, -a \sin u \sin v, a \cos v)
\]

is a basis (more precisely, a basis for the plane of tangent vectors). This local coordinate system is not necessarily orthogonal nor are the vectors necessarily of unit length, but this is not essential; we only need linear independence of the vectors.

We say a local coordinate system is right-handed if the determinant with respect to this natural basis is positive, otherwise we say that it is left-handed. We can reverse the orientation of a parametrization by interchanging the roles of any two variables of the parametrization (this is true for \(n\)-dimensional manifolds as well).

Another way to specify orientation for an \((n-1)\)-dimensional manifold \(S\) which is sitting (i.e. embedded) in \(\mathbb{R}^n\) is to specify a continuous normal vector field. For our example of a surface (2-dimensional manifold) embedded in \(\mathbb{R}^3\) we have that

\[
\vec{n} = \vec{n}(u, v) = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}
\]

\[
= (\cos u \cos v, \cos v \sin u, \sin v)
\]
is a continuous normal vector field. This method works in general since the cross product in $\mathbb{R}^n$ is an operation on $(n - 1)$ vectors which produces a vector orthogonal to each of these original vectors.
§2. Forms and the Exterior Derivative of a Form. In first quarter Calculus, one learns to integrate functions but the (Leibniz) notation, \( \int_{a}^{b} f(x)\,dx \), is initially puzzling. What is “\( dx \)”? The student is told that \( dx \) is a differential but this does not help much. Also puzzling is the equation \( dF = F'(x)\,dx \). In fact what one has here is the simplest situation of a 1-form, \( f(x)\,dx \), and the exterior derivative operator \( d \) applied to a function \( F \) (a function being a 0-form) in order to produce the 1-form \( dF \). If we let \( S \) be the closed interval \([a, b]\) oriented so that we start at \( x = a \) and end at \( x = b \), then \( \int_{a}^{b} F'(x)\,dx = \int_{S} dF \). Also the boundary of \( S \) is just the two points \( a \) (weighted with \(-1\)) and \( b \) (weighted with \(+1\)) so that \( F(b) - F(a) = \int_{\partial S} F \). The Fundamental Theorem of Calculus then states that

\[
\int_{S} dF = \int_{\partial S} F
\]

This is the simplest case of the abstract version of Stoke’s Theorem.

We define a 0-form on \( \mathbb{R}^n \) to be a differentiable function \( F = F(x_1, x_2, \ldots, x_n) \) on \( \mathbb{R}^n \). We define the exterior derivative \( dF \) of this 0-form to be the 1-form

\[
dF = \left( \frac{\partial F}{\partial x_1} \right) dx_1 + \left( \frac{\partial F}{\partial x_2} \right) dx_2 + \ldots + \left( \frac{\partial F}{\partial x_n} \right) dx_n
\]

We define a general 1-form \( \omega \) to be a formal linear combination

\[
\omega = \omega_1 dx_1 + \omega_2 dx_2 + \ldots + \omega_n dx_n
\]

where each \( \omega_i = \omega_i(x_1, x_2, \ldots, x_n) \) is a differentiable function for \( i = 0, 1, \ldots, n \). One can give a more precise definition but it would not shed much light without borrowing a significant amount of the theory of differential geometry.
At this point it is advisable to rewrite the line integrals in Section 1 in terms of forms. It is always possible to expand the vector field $\vec{A}$ in terms of its components, so we can write

$$\vec{A} = (A_1, A_2, \ldots, A_n)$$

$$= (A_1(x_1, x_2, \ldots, x_n), A_2(x_1, x_2, \ldots, x_n), \ldots, A_n(x_1, x_2, \ldots, x_n),$$

although we usually write $A_i$ rather than $A_i(x_1, x_2, \ldots, x_n)$ when the context is clear. Using our above expansion for $\vec{t}(s)$ We can then write

$$\int_C \vec{A} \cdot d\vec{s} = \int_C \vec{A} \cdot \vec{t}(s)ds = \int_C A_1dx_1 + A_2dx_2 + \ldots A_n dx_n$$

So, 0-forms and 1-forms are not that mysterious; we haven’t done much more than change notation. The important and non-trivial question is “how do we take the exterior derivative of a 1-form so that a 2-form is obtained?” Before we do this, we need to agree upon some rules concerning the algebra of forms. We are going to need a product (called the wedge product) for multiplying forms. This product should share the main properties of the cross product for vectors, but be more general. Certainly then we need to start with

1. If $\omega$ is a $j$-form and $\psi$ is a $k$-form then $\omega \wedge \psi$ is a $(j+k)$-form and

$$\omega \wedge \psi = -\psi \wedge \omega$$

2. Rule 1. above automatically forces

$$\omega \wedge \omega = 0$$

3. The wedge product $\wedge$ distributes over addition.

4. The exterior derivative operator $d$ is linear, that is,

$$d(\omega + \psi) = d\omega + d\psi$$

8
and for $\lambda \in \mathbb{R}$,

$$d(\lambda \omega) = \lambda d\omega$$

For example, let the dimension $n = 3$ and let $\omega$ be the 1-form above, so that

$$\omega = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$$

Using the above rules we can compute

$$d\omega = d\omega_1 \wedge dx_1 + d\omega_2 \wedge dx_2 + d\omega_3 \wedge dx_3$$

$$= \left( \left( \frac{\partial \omega_1}{\partial x_1} \right) dx_1 + \left( \frac{\partial \omega_1}{\partial x_2} \right) dx_2 + \left( \frac{\partial \omega_1}{\partial x_3} \right) dx_3 \right) \wedge dx_1 +$$

$$\left( \left( \frac{\partial \omega_2}{\partial x_1} \right) dx_1 + \left( \frac{\partial \omega_2}{\partial x_2} \right) dx_2 + \left( \frac{\partial \omega_2}{\partial x_3} \right) dx_3 \right) \wedge dx_2 +$$

$$\left( \left( \frac{\partial \omega_3}{\partial x_1} \right) dx_1 + \left( \frac{\partial \omega_3}{\partial x_2} \right) dx_2 + \left( \frac{\partial \omega_3}{\partial x_3} \right) dx_3 \right) \wedge dx_3 =$$

$$\left( \frac{\partial \omega_3}{\partial x_2} - \frac{\partial \omega_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial \omega_1}{\partial x_3} - \frac{\partial \omega_3}{\partial x_1} \right) dx_3 \wedge dx_1 +$$

$$\left( \frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

$$= \text{det} \begin{pmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
\omega_1 & \omega_2 & \omega_3
\end{pmatrix}$$

since $dx_i \wedge dx_i = 0$ for $i = 1, 2, 3$ and $-dx_1 \wedge dx_3 = dx_3 \wedge dx_1$. But this looks suspiciously like the curl.